Evaluating Failure Rate of Fault-Tolerant Multistage Interconnection Networks Using Weibull Life Distribution

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Abstract

Fault-tolerant multistage interconnection networks (MINs) play a vital role in the performance of multiprocessor systems where reliability evaluation becomes one of the main concerns in analyzing these networks properly. In many cases, the primary objective in system reliability analysis is to compute a failure distribution of the entire system according to that of its components. However, since the problem is known to be NP-hard, in none of the previous efforts, the precise evaluation of the system failure rate has been performed. Therefore, our goal is to investigate this parameter for different fault-tolerant MINs using Weibull life distribution that is one of the most commonly used distributions in reliability. In this paper, four important groups of fault-tolerant MINs will be examined to find the best fault-tolerance techniques in terms of failure rate; (1) Extra-stage MINs, (2) Parallel MINs, (3) Rearrangeable non-blocking MINs, and (4) Replicated MINs. This paper comprehensively analyzes all perspectives of the reliability (terminal, broadcast, and network reliability). Moreover, in this study, all reliability equations are calculated for different network sizes.

Keywords: Parallel computers, Multistage Interconnection networks, Failure rate, Weibull life distribution, Pars network.

Acronyms and abbreviations

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Notations

- $N$: Number of sources and destinations (network size)
- $L$: Number of network layers
- $R(t)$: Time-dependent reliability
- $t$: Operating time in the useful life period
- $\lambda(t)$: The failure rate
- $f(t)$: Probability density function
- $F(t)$: Cumulative density function
- $\eta$: Scale parameter
- $\beta$: Shape parameter
- $\gamma$: Location parameter
- $r(t)$: Reliability of $2 \times 2$ switching elements
- $TR(t)$: Terminal reliability
- $T\lambda(t)$: Terminal failure rate
- $BR(t)$: Broadcast reliability
- $B\lambda(t)$: Broadcast failure rate
- $NR(t)$: Network reliability
- $N\lambda(t)$: Network failure rate

1. Introduction

The main approach for improving the fault-tolerance of multistage interconnection networks (MINs) is to create redundant paths between each source-destination pair. So far, numerous network topologies have been proposed for this purpose. However, the question here is which one can be an efficient solution to tackle this problem? Reliability is a significant metric of performance among network technologies [1] such as lifeline networks (e.g. electrical and gas networks) [2, 3, 4], wireless mobile ad hoc networks (MANETs) [5, 6, 7], wireless mesh networks [8-11], wireless sensor networks [12-14], sensors based on nanowire networks [15], social networks [16], stochastic-flow manufacturing networks (SMNs) [17], and interconnection networks [18-27]. Following are some of the latest

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reliability of bi-directional data vortex has been analyzed. The comparison of proposed bidirectional architecture (i.e. a MIN) with packets routed both in forward and backward directions, is discussed in [37]. The terminal proposed designs obtains higher reliability values and fault tolerance. A bi-directional data vortex interconnection time-independent terminal and network reliability analysis results with some existing MINs demonstrated that the and RIN-2) are introduced in Ref. [36] as two new architecture designs of fault tolerant MINs. A comparison of the used to evaluate the reliability and fault tolerance of existing HCNs. The reliable interconnection networks (RIN -1 encompasses the fault tolerance and reliability characteristics of hypercube multi-computer networks (HCNs). Specifically, this work covers three basic aspects: task-based reliability, fault-tolerant routing, and communication in interconnection networks are extra connectivity determination and faulty networks’ structure analysis. Consequently, in [34], it was shown that for $n \geq 4$ there is a large connected component in an incomplete $n$-dimensional bijective connection network (in brief, BC network), and the remaining small components have at most $h$ vertices in sum provided that the total number of faulty vertices is strictly smaller than its $h$-extra connectivity ($0 \leq h \leq n - 4$). The $h$-extra connectivity of an $n$-dimensional BC network for $0 \leq h \leq n - 4$ also achieved. Furthermore, applying these results to an $n$-dimensional hypercube, Möbius cube, twisted cube, and crossed cube, the results on the structure of these faulty networks and their $h$-extra connectivity ($0 \leq h \leq n - 4$) are obtained. These results demonstrated that the reliability of hypercubes is maintained by Möbius cubes, twisted cubes, and crossed cubes. Having their diameters is close to a half of that of hypercube’s, these networks can potentially be suitable alternatives to hypercubes as the topology of next generation interconnection network. Ref. [35] encompasses the fault tolerance and reliability characteristics of hypercube multi-computer networks (HCNs). Specifically, this work covers three basic aspects: task-based reliability, fault-tolerant routing, and communication in faulty HCNs. It introduces the special HC architecture, analyzes its reliability and evaluates its fault tolerance. The performed analysis in this work assists HCN designers to make knowingly decisions about approaches that can be used to evaluate the reliability and fault tolerance of existing HCNs. The reliable interconnection networks (RIN-1 and RIN-2) are introduced in Ref. [36] as two new architecture designs of fault tolerant MINs. A comparison of the time-independent terminal and network reliability analysis results with some existing MINs demonstrated that the proposed designs obtains higher reliability values and fault tolerance. A bi-directional data vortex interconnection network (i.e. a MIN) with packets routed both in forward and backward directions, is discussed in [37]. The terminal reliability of bi-directional data vortex has been analyzed. The comparison of proposed bidirectional architecture with two parallel connected data vortex networks in terms of terminal reliability showed that the terminal reliability of the bi directional data vortex is a bit higher. To solve the network reliability optimization problem of the interconnection networks in case of the failure of nodes and links, a new technique based upon artificial neural network is presented in [38]. Two significant performance parameters of an interconnection network that should be placed at the stage of layout design, which is necessary for an appropriate trade-off between them, are reliability and
cost. Therefore, [39] proposed a new approach based on dynamic programming for solving a type of layout optimization problem, where new nodes and their corresponding links are added to an existing interconnection network for maximization of network reliability and minimization of cost. In Ref. [40], two new designs of 4-disjoint paths multistage interconnection networks, called 4-disjoint gamma interconnection networks (4DGIN-1 and 4DGIN-2) are proposed. In this work, the time-independent terminal reliability for different tag values is computed by generating all possible paths for each source and destination pair of the network and then evaluating terminal reliability from the minimal paths. The higher terminal reliability value of the 4DGINs for all the tag values ensure more robust performance compared to GIN.

The results obtained from mentioned above and other previous studies (I.e., [18-26, 41-43]) are very valuable and useful to understand fault-tolerant MINs. However, there are some gaps in previous works, which do not make them as comprehensive studies: (1) The previous works focused mainly on two methods: increasing the number of stages [18-21, 24-26] and using several MINs in parallel [20, 22, 23, 41, 44]. While other methods such as concatenating a single-path MIN with its reverse MIN [44, 45] and using replicated MINs [20, 46] are also very important for evaluation. (2) Often, previous works focused on three parameters of reliability: time-independent reliability, time-dependent reliability, and MTTF. However, one basic objective in reliability analysis of fault-tolerant systems is to obtain a failure distribution (also called the hazard rate) of the entire system based on the failure distribution of its components. (3) There are three metrics to assess the reliability of MINs: terminal, broadcast, and network reliability. However, few previous works have examined all three metrics (4) Most of the works are based on approximated solutions which consider bounds for reliability, especially for large network sizes due to the difficulties of reliability analysis in fault-tolerant MINs.

Given the above arguments, our goal is to provide a more comprehensive investigation compared with previous works that could address these above issues. In summary, in this paper, we will analyze the failure rate of four different types of fault-tolerant networks: (1) Increasing the number of stages. (2) Using several MINs in parallel. (3) Concatenating a single-path MIN with its reverse MIN. (4) Using replicated MINs. For this purpose, we selected four fault-tolerant networks for evaluation, which all of them have recently attracted the attention of many researchers. These four types of fault-tolerant MINs are as follows: Extra-stage shuffle-exchange network (SEN+) on behalf of the Group 1 [19, 21, 25], Pars network on behalf of the Group 2 [41], Benes network (a rearrangeable non-blocking MIN) on behalf of the Group 3 [44, 45], and finally two-layer replicated MIN on behalf of the Group 4 [20, 46].

The extra-stage shuffle-exchange network (SEN+) is a SEN with one additional stage. SEN+ of size $8 \times 8$ is shown in Fig. 1(a). A SEN+ of size $N \times N$ is comprised of $(\log_2 N + 1)$ stages and each stage is composed of $\frac{N}{2}$ switching elements of size $2 \times 2$. In this network, there are two paths between each source-destination pair. The network complexity of an $N \times N$ SEN+ is $\frac{N}{2} (\log_2 N + 1)$.

Fig. 1(b) shows an $8 \times 8$ Pars network. A Pars network of size $N \times N$ consists of $(\log_2 N - 1)$ stages of $\left(\frac{3N}{4}\right)$ switches. All switching elements are of size $2 \times 2$ which this small size will reduce costs. There is one $4 \times 1$ multiplexer for each input link in a stage 1 and one $1 \times 2$ demultiplexer for each output link of a switch in stage $(\log_2 N - 1)$. Therefore, a Pars network of size $N \times N$ has $\left(\frac{3N}{2}\right)$ multiplexers in the input stage and $\left(\frac{3N}{2}\right)$ demultiplexers in the output stages. The network complexity of an $N \times N$ Pars network is $\left(\frac{3N}{2}\right) (3 + ((\log_2 N) - 1))$. The Pars network provides a total number of six paths between each source-destination pair that this redundancy in network paths can improve the ability of fault tolerant network.

Fig. 1. (a) $8 \times 8$ SEN+, (b) $8 \times 8$ Pars network, (c) $8 \times 8$ Benes network, and (d) $8 \times 8$ two-layer replicated MIN.

A Benes network of size $8 \times 8$ is shown in Fig. 1(c). A Benes network of size $N \times N$ consists of $(2 \log_2 N - 1)$ stages and each stage is composed of $\frac{N}{2}$ switching elements of size $2 \times 2$. The network complexity of an $N \times N$ Benes network is $\frac{N}{2} (2 \log_2 N - 1)$. The Benes network is best-known example of a rearrangeable non-blocking MIN. In these networks, any input port can be connected to any free output port. However, the existing connections may require rearrangement of paths. These networks also have multiple paths between every source and destination.

Fig. 1(d) shows the architecture of an $8 \times 8$ replicated MIN consisting of two layers $(L=2)$ in a three dimensional view. In this paper, our focus is on the two-layer replicated MIN. An L-layer replicated MIN of size $N \times N$ consists of $(\log_2 N)$ stages of $\left(\frac{L \times N}{2}\right)$ switching elements. Typically, all switching elements are of size $2 \times 2$. In addition, a L-layer
replicated MIN of size $N \times N$ has $N$ demultiplexers of size $1 \times L$ in the input stage and $N$ multiplexers of size $L \times 1$ in the output stage. The network complexity of an $N \times N$ L-layer replicated MIN is $\left(\frac{L}{2}\right)(1 + (\log_2 N))$.

The rest of the paper is organized as follows: Contribution of the paper will be discussed in Section 2. In Section 3, terminal, broadcast, and network failure rate will be analyzed. Finally, Section 4 concludes the paper.

2. Contribution

One of the most important contributions of this paper is to evaluate the failure rate of fault-tolerant MINs, which has received less attention in the past. On the other hand, to the best of our knowledge, in none of the previous works an analysis of failure rates of all reliability measurements on the MINs (i.e. terminal, broadcast, and network) has been carried out. Hence, this paper evaluates the failure rate of the MINs for all reliability measurements for the first time. The next issue is related to the network size. We need to obtain the reliability equations for larger network sizes in order to get closer to reality and large-scale systems. But the problem here is the increased number of components and the complexity of relations among which in terms of reliability. However, in this paper, all reliability and failure rate equations for all three measures of reliability namely terminal, broadcast, and network will be calculated for general size $N \times N$. On the other hand, it should be noted that previous works has often used the exponential distribution for the times-to-failures of the components (switching elements) [23, 25, 41, 42]. When a component is subject only to functional failures that occur at random intervals, and the expected number of failures is the same for equally long periods of time, its probability density function and its reliability can be defined by the exponential distribution. Although the determination of equipment reliability and corresponding system reliability during the period of the equipment’s useful life period is based on the exponential failure distribution, the failure rate of the equipment may not be constant throughout the period of its use or operation. In most engineering installations, particularly with the integration of complex systems, the purpose of determining equipment criticality, or combinations of critical equipment, is predominantly to assess the times to wear-out failures, rather than to assess the times to chance or random failures. In such cases, the exponential failure distribution does not apply, and it becomes necessary to substitute a general failure distribution, such as the Weibull distribution. The Weibull distribution is particularly useful because it can be applied to all three of the phases of the hazard rate curve, which is also called the equipment ‘life characteristic curve’ [47]. Therefore, in this paper, suppose that the times-to-failures of the components (switching elements) are described with Weibull distribution. Here, it should be noted that the reliability of a certain class of MINs, called shuffle-exchange networks (SEns) was assessed using the Weibull distribution in [24]. However, this analysis is only based on a special class of MINs. On the other hand, it is only focused on network (all-terminal) reliability. In addition, failure rate parameter has not been analyzed in this study.

Therefore, in this paper, our focus is on reliability evaluation of four important methods proposed to improve fault-tolerance in MINs. The results of this study are very important in terms of finding the best fault-tolerance techniques in the field of MINs. Also, it should be noted that the methodology used is to model the networks in series-parallel or complex (non-series-parallel) RBDs instead of using the estimated upper and lower bounds RBDs. Therefore, this approach can provide accurate information about reliability of the networks compared with previous works. Moreover, the methodology is applicable for evaluating the failure rate of other MINs, even with more complex structures.

3. Failure rate evaluation of the MINs

Mathematically, reliability $R(t)$ is the probability that a system will be successful in the interval from time 0 to time $t$. On the other hand, the failure rate $\lambda(t)$ is the instantaneous failure rate, and is defined by [27, 48].

$$\lambda(t) = \lim_{\Delta t \to 0} \frac{R(t) - R(t + \Delta t)}{\Delta t R(t)} = \frac{1}{R(t)} \left( - \frac{d(R(t))}{dt} \right)$$

(1)

As earlier discussed, it will be assumed that the times-to-failures of the components (switching elements) are described with Weibull life distribution.

The reliability function for the 3-parameter Weibull distribution is given by:

$$R(t) = e^{-\left(\frac{t}{\eta}\right)^\beta}, t \geq \gamma \geq 0, \beta > 0, \eta > 0$$

(2)
where $\eta$ and $\beta$ are known as the scale and shape parameters, respectively, and $\gamma$ is known as the location parameter. These parameters are always positive. By using different parameters, this distribution can follow the exponential distribution, the normal distribution, etc.

Therefore, The Weibull failure rate function, $\lambda(t)$, is given by:

$$\lambda(t) = \frac{1}{R(t)} \left( \frac{d[R(t)]}{dt} \right) = \frac{\beta}{\eta} \left( \frac{t}{\eta} \right)^{\beta-1}, \quad t \geq 0, \beta > 0, \eta > 0 \quad (3)$$

It can be shown that the failure rate function decreases for $\beta < 1$, increases for $\beta > 1$, and is constant when $\beta = 1$. Note that the Rayleigh and exponential distributions are special cases of the Weibull distribution at $\beta = 2$, $\gamma = 0$, and $\beta = 1$, $\gamma = 0$, respectively.

3.1. Reliability and failure rate analysis

According to equation (1), it is necessary to calculate the reliability in order to calculate the failure rate. Therefore, first, the reliability equation of each of the networks will be calculated using reliability block diagrams (RBDs), then the failure rate of the networks will be calculated according to the reliability equations.

In this paper, we will use the switch fault model for reliability analysis. Therefore, it will be assumed that each switching component such as switching element (SE), multiplexer (MUX), and demultiplexer (DMUX) may fail. We assume that the $r(t)$ is the probability of a 2×2 switching element (SE2×2) being operational. Also, given the number of gates in switching elements of different sizes, their operational probability can be computed based on $r(t)$ [22, 23, 41]. It is also assumed that the hardware complexity of a component is directly proportional to the number of gates [23, 41].

3.1.1. Terminal failure rate

Terminal reliability is an important reliability measure in most complex networks. It refers to the probability of successful communication between each source-destination pairs in the network according to the topology of the network. Therefore, in case of regular MINs such as SEN+, Pars, Benes, and replicated MIN, the terminal reliability can be calculated using RBDs by considering a source-destination pair.

According to the concept of terminal reliability as well as with respect to the topology of the networks, terminal RBDs for each network are shown in Figs. 2, 3, 6, and 7. In these RBDs, blocks corresponding to switching elements of size $c\times c$ are shown with SEc and blocks corresponding to multiplexers of size $n\times1$ and demultiplexers of size $1\times n$ are shown with MXn and DMn, respectively.

According to Fig. 2, the terminal reliability ($TR(t)$) of SEN+ is given by:

$$TR_{SEN+}(t) = r(t)^2 \left( 1 - \left( 1 - r(t)^{log_2 N-1} \right)^2 \right) \quad (4)$$

Some RBDs cannot be properly defined in one of three models of series, parallel, and series-parallel systems in terms of reliability, therefore, these are known as complex systems. The diagram shown in Fig. 3 is a complex (non-series-parallel) RBD. One solution to determine the reliability of a complex system is to use the decomposition method that is an application of the law of total probability. In this method, first, one of the components is selected as a key component and then the system reliability is calculated in two modes: when the key component is activated and when the key component has been failed. Finally, these two probabilities are combined to achieve the reliability of the whole system.

By calculating the reliability of the series blocks in the RBD of Fig. 3, the diagram can be plotted in Fig. 4. In Fig. 4, the reliability of each block is written on its side. The RBD is from the complex type, therefore, to analyze it, the decomposition method can be used. For this purpose, we need to pick one block as the key block. The key block is highlighted in Fig. 4. Therefore, we have:

$$TR_{Pars} = R_{key}(R_{pars}|R_{key}) + R_{key}R_{pars}|R_{key} \quad (5)$$

On the other hand, we have:

$$R_{key}(R_{pars}|R_{key}) = r(t)^2 \left[ 1 - \left( 1 - r(t)^{log_2 N-1} \right)^2 \right] \left( 1 - r(t)^{log_2 N-1} \left( 1 - r(t)^2 \right)^2 \right) \quad (6)$$

To calculate the second term $R_{key}(R_{pars}|R_{key})$ of the equation (5), the RBD in Fig. 4 can be considered in Fig. 5, that again one block is highlighted as a key block. According to this RBD, we have:

$$R_{key}(R_{pars}|R_{key}) = (1 - r(t)^2) \left[ R(t)^2 \left( 1 - \left( 1 - r(t)^{log_2 N-1} \right)^2 \right) \left( 1 - r(t)^{log_2 N-1} \right)^2 \right] + (1 - r(t)^2) \left( R(t)^2 \left( 1 - \left( 1 - r(t)^{log_2 N-1} \right)^2 \right) \right) \quad (7)$$
Therefore, considering equations (5) to (7), the terminal reliability of N×N Pars network is given by:

\[ TR_{\text{Pars}}(t) = r(t)^2 \left[ 1 - \left( 1 - r(t) \left( \log_2 N \right)^{-2} \right) \right] \left( 1 - \left( 1 - r(t) \left( \log_2 N \right)^{-2} \right)^2 \right) \]

The terminal RBDs for 8×8 and 16×16 Benes network are shown in Figs. 6(a) and 6(b), respectively. Regarding Fig. 6(a), terminal reliability of 8×8 Benes network is calculated by the equation (9):

\[ TR_{8\times8\text{Benes}}(t) = r(t)^2 \left[ 1 - \left( 1 - r(t) \left( \log_2 8 \right)^{-2} \right)^2 \right] \]

Fig. 5. Terminal reliability RBD of N×N Pars network for equation \( R_{\text{key}}(R_{\text{Pars}} | R_{\text{key}}) \).

Fig. 6. Terminal reliability RBD for (a) 8×8 Benes network and (b) 16×16 Benes network.

Also, according to the terminal reliability RBD of 16×16 Benes network in Fig. 6(b), the terminal reliability of an N×N Benes network is given by:

\[ TR_{N\timesN\text{Benes}}(t) = r(t)^2 \left[ 1 - \left( 1 - r(t) \left( \log_2 N \right)^{-2} \right)^3 \right] \]

Fig. 7. Terminal reliability RBD for N×N two-layer replicated MIN.

Similarly, according to Fig. 7, we have:

\[ TR_{\text{two-layer}}(t) = r(t) \left( 1 - r(t) \left( \log_2 N \right)^{-2} \right)^3 \]

As previously mentioned, it is assumed that times-to-failures of switching components are described with Weibull life distribution. Therefore, according to equations (1), (2), and (4) to (11), the terminal failure rate \( \lambda(t) \) of the networks is computed as follows:

\[ \lambda(t) = \frac{1}{\log_2(N-2)} \left( \frac{1}{e^{\frac{\beta}{\eta}} - 1} \right)^2 \sum_{i=1}^{E_1} \left[ \frac{1}{e^{\left( \frac{\beta}{\eta} \right)^i} - 1} \right] \]

The terminal failure rate of Pars network can be expressed as follows:

\[ \lambda_{\text{Pars}}(t) = \frac{1}{\log_2(N-2)} \left( \frac{1}{e^{\left( \frac{\beta}{\eta} \right)^1} - 1} \right)^2 \sum_{i=1}^{E_1} \left[ \frac{1}{e^{\left( \frac{\beta}{\eta} \right)^i} - 1} \right] \]

In equation (13), each of the equations E1 to E10 are calculated as equations (34) through (43) in the Appendix.

Also, for the Benes network and two-layer MIN, we have:

\[ \lambda_{\text{Benes}}(t) = \frac{1}{\log_2(N-2)} \left( \frac{1}{e^{\left( \frac{\beta}{\eta} \right)^1} - 1} \right)^2 \sum_{i=1}^{E_1} \left[ \frac{1}{e^{\left( \frac{\beta}{\eta} \right)^i} - 1} \right] \]
\[ T_{\text{two-layer}}(t) = \frac{e^{-(\frac{N+1}{2})\beta t}}{\eta e^{(\frac{N}{2})\beta t}} \left( 1 - e^{-e^{-\left(\frac{N-2}{2}\right)\beta t}} \right) \left( 1 - e^{-\left(\frac{N-2}{2}\right)\beta t} \right)^{\frac{N+2}{2}} \right)^{2} \quad (15) \]

It should be noted that in equation (14), the failure rate is calculated for Benes network of size 8×8. However, considering the equations (1), (9), and (10), we can easily calculate the failure rate for any other desired network size.

3.1.2. Broadcast failure rate

In order to analyze the failure rate of point-broadcast, we first need to calculate the broadcast reliability equations of the networks. Broadcast reliability is defined as the probability of successfully establishing a connection between a source and all destinations in the network. According to this definition, the broadcast reliability can be calculated by considering one source and all destinations in the network.

The broadcast reliability RBD of the N×N SEN+ is shown in Fig. 8. According to this figure, the broadcast reliability (BR(t)) of SEN+ is given by:

\[ BR_{\text{SEN+}}(t) = \frac{1}{2} \left( 1 - \left( 1 - r(t) \right)^{2} \right) \quad (16) \]

Fig. 8. Broadcast reliability RBD for N×N SEN+ [19].

Fig. 9. Broadcast reliability RBD for N×N Pars network.

Also, Fig. 9 can be seen as a series RBD consisting of two blocks, each of these blocks is a non-series-parallel RBD. It should be noted that the reliability of these non-series-parallel RBDs can be computed similar to reliability analysis of the RBD shown in Fig. 3. Therefore, according to Fig. 9, we have:

\[ BR_{\text{Pars}}(t) = \left( r(t)^{2} \left( 1 - \left( 1 - \left( 1 - r(t) \right)^{2} \right) \right) \right) + \left( 1 - r(t)^{2} \left( 1 - \left( 1 - r(t) \right)^{2} \right) \right) \quad (17) \]

Similarly, according to Figs. 10 and 11, the broadcast failure rate of Benes network and two-layer network are calculated as:

\[ BR_{\text{Benes}}(t) = \left( 1 - r(t)^{2} \left( 1 - \left( 1 - r(t) \right)^{2} \right) \right) \quad (18) \]

\[ BR_{\text{Benes}}(t) = \left( 1 - r(t)^{2} \left( 1 - \left( 1 - r(t) \right)^{2} \right) \right) \quad (19) \]

\[ BR_{\text{two-layer}}(t) = \left( 1 - r(t)^{2} \left( 1 - \left( 1 - r(t) \right)^{2} \right) \right) \quad (20) \]

Fig. 10. Broadcast reliability RBD for (a) 8×8 Benes network and (b) 16×16 Benes network.

Fig. 11. Broadcast reliability RBD for N×N two-layer replicated MIN.

Therefore, according to equations (1), (2), and equations (16) to (20), the terminal failure rate (Br(t)) of the networks is given by:

\[ Br_{\text{SEN+}}(t) = \frac{\beta (N+1)^{\beta-1} e^{-\left(\frac{N+1}{2}\right)\beta t}}{2} \left( 1 - e^{-e^{-\left(\frac{N-2}{2}\right)\beta t}} \right) \left( 1 - e^{-e^{-\left(\frac{N-2}{2}\right)\beta t}} \right)^{\frac{N+2}{2}} \right) \quad (21) \]

The broadcast failure rate of Pars network can be expressed as follows:
In equation (22), each of the equations E1 to E8 are calculated as equations (44) through (51) in the Appendix.

\[
B\lambda_{pars}(t) = \frac{2 \left( -E_2 + \frac{E_2 + E_3}{2} \right) \left( \frac{1}{E_2 + E_3} \right) \left( \frac{1}{2} \right) \left( \frac{1}{E_2 + E_3} \right)}{E_2 + \left( \frac{1}{2} \right) \left( \frac{1}{E_2 + E_3} \right) \left( \frac{1}{E_2 + E_3} \right)}
\]  

(22)

\[
B\lambda_{apxs\, Bene}es(t) = \frac{\left( \frac{1}{E_2 + E_3} \right) \left( \frac{1}{E_2 + E_3} \right) \left( \frac{1}{E_2 + E_3} \right) \left( \frac{1}{E_2 + E_3} \right)}{1 - \left( \frac{1}{E_2 + E_3} \right) \left( \frac{1}{E_2 + E_3} \right) \left( \frac{1}{E_2 + E_3} \right) \left( \frac{1}{E_2 + E_3} \right)}
\]  

(23)

\[
B\lambda_{two\,-\,layer}(t) = \frac{\left( \frac{1}{E_2 + E_3} \right) \left( \frac{1}{E_2 + E_3} \right) \left( \frac{1}{E_2 + E_3} \right) \left( \frac{1}{E_2 + E_3} \right)}{1 - \left( \frac{1}{E_2 + E_3} \right) \left( \frac{1}{E_2 + E_3} \right) \left( \frac{1}{E_2 + E_3} \right) \left( \frac{1}{E_2 + E_3} \right)}
\]  

(24)

### 3.1.3. Network failure rate

Network reliability refers to the probability of establishing successful communication between all source and destinations, taking into account the topology of the network. Therefore, all source and destination nodes must be considered in calculating the network reliability. According to this definition, the network reliability RBDs of the networks are shown in Figs. 12, 13, 14, and 15. Considering Fig. 12, we have [19]:

\[
NR_{SEN+}(t) = r(t)^N \left( 1 - (1 - r(t))^N \right)^2 \left( 1 - \left( 1 - (1 - r(t))^N \right)^2 \right)
\]  

(25)

Also, Fig. 13 can be considered as a series system consisting of two blocks. One of these blocks is a non-series-parallel system and another one is a series-parallel. It should be noted that the reliability of the non-series-parallel RBDs can be calculated similar to reliability of the RBD depicted in Fig. 3. Therefore, according to Fig. 13, we have:

\[
NR_{pars}(t) = \left( r(t)^N \right)^2 \left( 1 - \left( 1 - (1 - r(t))^N \right)^2 \right) \left( 1 - \left( 1 - (1 - r(t))^N \right)^2 \right) + \left( 1 - r(t)^N \right) \left( 1 - \left( 1 - (1 - r(t))^N \right)^2 \right)
\]  

(26)

Fig. 12. Network reliability RBD for N×N SEN+ [19].

Fig. 13. Network reliability RBD for N×N Pars network.

Moreover, according to Figs. 14 and 15, we have [49]:

\[
NR_{two\,-\,layer}(t) = r(t)^N \left( 1 - \left( 1 - r(t)^N \right)^2 \right)
\]  

(27)

\[
NR_{apxs\, Bene}es(t) = r(t)^N \left( 1 - (1 - r(t))^N \right)^2 \left( 1 - (1 - r(t)^N \left( 1 - (1 - r(t))^2 \right)^2 \right)
\]  

(28)

\[
NR_{two\,-\,layer}(t) = r(t)^N \left( 1 - (1 - r(t))^2 \right) \left( 1 - \frac{NR_{apxs\, Bene}es(t)^2}{r(t)^N} \right)
\]  

(29)

Fig. 14. Network reliability RBD for N×N two-layer MIN.

Fig. 15. Network reliability RBD for (a) 8×8 Bene network and (b) 16×16 Bene network.
In this paper, it is assumed that the times-to-failures of the components (switching elements) are described with Weibull distribution. Therefore, the network failure rate of the networks is given by the following equations:

$$N\lambda_{SEN+}(t) = \frac{e^{N\left(\frac{\gamma(T)}{\eta}\right)^\beta}}{\beta} \left(\frac{1}{e^{\left(\frac{\gamma(T)}{\eta}\right)^\beta}} + 1\right)^2 \left(\frac{1}{\frac{N\left(\log_2(N/2)\right)^\beta}{e^{\left(\frac{\gamma(T)}{\eta}\right)^\beta}} + 1}\right)^{-1} (E_1 + E_2 + E_3)$$

(30)

In equation (30), equations E1 to E3 are computed as equations (52) through (54) in the Appendix.

$$N\lambda_{theta}(t) = \left(\frac{1}{1 - e^{-N\left(\log_2(N/2)\right)^\beta}} + 1\right) \left(\frac{N\left(\log_2(N/2)\right)^\beta}{e^{\left(\frac{\gamma(T)}{\eta}\right)^\beta}} + 1\right)^2 (E_1 + E_2 + E_3) + 1 + E_6 + E_7$$

(31)

In equation (31), equations E1 to E12 are computed as equations (55) through (66) in the Appendix.

$$N\lambda_{two-layer}(t) = \frac{e^{N\left(\frac{\gamma(T)}{\eta}\right)^\beta}}{\beta} \left(\frac{1}{e^{\left(\frac{\gamma(T)}{\eta}\right)^\beta}} + 1\right)^2 \left(\frac{1}{\frac{N\left(\log_2(N/2)\right)^\beta}{e^{\left(\frac{\gamma(T)}{\eta}\right)^\beta}} + 1}\right)^{-1} (E_1 + E_2 + E_3)$$

(32)

$$N\lambda_{vibRemeq}(t) = \frac{e^{N\left(\frac{\gamma(T)}{\eta}\right)^\beta}}{\beta} \left(\frac{1}{e^{\left(\frac{\gamma(T)}{\eta}\right)^\beta}} + 1\right)^2 \left(\frac{1}{\frac{N\left(\log_2(N/2)\right)^\beta}{e^{\left(\frac{\gamma(T)}{\eta}\right)^\beta}} + 1}\right)^{-1} (E_1 + E_2 + E_3)$$

(33)

In equation (33), equations E1 to E3 are computed as equations (67) through (69) in the Appendix.

3.2. Numerical results

To obtain numerical results from the equations resulted in the previous section, the value of the variable parameters in the equations must be specified.

The slope or shape parameter $\beta$ is usually chosen in the following ranges: $\beta < 1$, $\beta = 1$, and $\beta > 1$ [47, 50]. A value of $\beta < 1$ indicates that the failure rate (hazard function) decreases over time. This happens if there is significant "infant mortality", or defective items failing early and the failure rate decreasing over time as the defective items are weeded out of the population. Also, a value of $\beta = 1$ indicates that the failure rate is constant over time. This might suggest random external events are causing mortality, or failure. Finally, a value of $\beta > 1$ indicates that the failure rate increases with time. This happens if there is an "aging" process, or parts that are more likely to fail as time goes on. In this paper, we'll examine all of these cases by selecting the $\beta$ as 0.5, 1, and 2.

The location parameter $\gamma$ is also known as the threshold parameter, or, in life testing applications, as the guarantee time, since failure cannot occur until $t$ exceeds $\gamma$. In many cases, it is usually assumed that the $\gamma$ is zero. When $\gamma$ is zero the three-parameter Weibull distribution specializes to much more widely employed two-parameter version [50, 51]. To obtain numerical results, we will assume that $\gamma = 0$ so that the model becomes the two-parameter Weibull distribution.

In case of characteristic life or scale parameter $\eta$, to determine the appropriate value for this parameter it can be noted in the probability density function (pdf). In fact, the pdf represents the relative frequency of failure times as a function of time. The scale parameter is the 63.2 percentile of the data. A scale of 20, for example, indicates that 63.2% of the equipment will fail in the first 20 hours after the threshold time. For a better understanding, Fig. 16 demonstrates the effect of the scale parameter, $\eta$, on the Weibull pdf for two samples $\beta=1$ and $\beta=2.5$. For example, consider the $\beta=2.5$, as it is visible in Fig. 16, in the event that $\eta=50$, the relative frequency of failure times is higher in early times in comparison with the case $\eta=100$. In other words, by increasing the $\eta$, the amplitude of pdf curve also decreases. However, by increasing the $\eta$, the relative frequency of failure times covered more times.
In the field of interconnection networks, it is usually assumed that the relative frequency of failure times will be more distributed at the high operating times. Because of this issue and considering Fig. 16 we will discuss an operating time range (τ) from 100 to 800 hours and η=6000 which is a reasonable estimate for this study. In addition, we will analyze the results for different network sizes of 8 to 128.

Fig. 16. The Weibull pdf for different values of η [52].

### 3.2.1. Numerical results of the terminal failure rate analysis

According to equations (16) through (19), the results of the terminal failure rate analysis of 8×8 and 128×128 networks as a function of time for β =0.5, 1, and 2 are summarized in Tables 1 to 3, respectively.

According to the results in Tables 1 to 3, it is evident that Pars network has an obvious advantage compared to the other networks in terms of terminal failure rate for all times. Also, the supremacy of the Pars exists in all scenarios for the parameter β (0.5, 1, and 2) and for both network sizes 8 and 128. Since Pars network is a MIN designed using several MINs in parallel, these results suggest that use of the parallel MINs has a higher potential than the other methods of increasing the number of stages, concatenating a MIN with its reverse MIN, and using replicated MINs in terms of providing a lower terminal failure rate.

#### Table 1. Terminal failure rate as a function of time for β =0.5.

<table>
<thead>
<tr>
<th>Time (Hr)</th>
<th>SEN+</th>
<th>Pars network</th>
<th>Benes network</th>
<th>Two-layer MIN</th>
<th>SEN+</th>
<th>Pars network</th>
<th>Benes network</th>
<th>Two-layer MIN</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.000770</td>
<td>0.000281</td>
<td>0.001843</td>
<td>0.001587</td>
<td>0.004004</td>
<td>0.001941</td>
<td>0.002326</td>
<td>0.004016</td>
</tr>
<tr>
<td>200</td>
<td>0.001340</td>
<td>0.000338</td>
<td>0.001425</td>
<td>0.001269</td>
<td>0.003102</td>
<td>0.001887</td>
<td>0.002132</td>
<td>0.003134</td>
</tr>
<tr>
<td>300</td>
<td>0.001140</td>
<td>0.000365</td>
<td>0.001230</td>
<td>0.001107</td>
<td>0.002645</td>
<td>0.001787</td>
<td>0.002088</td>
<td>0.002677</td>
</tr>
<tr>
<td>400</td>
<td>0.001016</td>
<td>0.000379</td>
<td>0.001109</td>
<td>0.001001</td>
<td>0.002352</td>
<td>0.001691</td>
<td>0.00207</td>
<td>0.002380</td>
</tr>
<tr>
<td>500</td>
<td>0.000929</td>
<td>0.000386</td>
<td>0.001023</td>
<td>0.000924</td>
<td>0.002141</td>
<td>0.001606</td>
<td>0.002054</td>
<td>0.002166</td>
</tr>
<tr>
<td>600</td>
<td>0.000863</td>
<td>0.000389</td>
<td>0.000958</td>
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<td>0.001980</td>
<td>0.001531</td>
<td>0.002035</td>
<td>0.002001</td>
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<tr>
<td>700</td>
<td>0.000811</td>
<td>0.000390</td>
<td>0.000905</td>
<td>0.000816</td>
<td>0.001851</td>
<td>0.001464</td>
<td>0.002012</td>
<td>0.001870</td>
</tr>
<tr>
<td>800</td>
<td>0.000768</td>
<td>0.000389</td>
<td>0.000862</td>
<td>0.000775</td>
<td>0.001745</td>
<td>0.001405</td>
<td>0.001986</td>
<td>0.001761</td>
</tr>
</tbody>
</table>

#### Table 2. Terminal failure rate as a function of time for β =1.

<table>
<thead>
<tr>
<th>Time (Hr)</th>
<th>SEN+</th>
<th>Pars network</th>
<th>Benes network</th>
<th>Two-layer MIN</th>
<th>SEN+</th>
<th>Pars network</th>
<th>Benes network</th>
<th>Two-layer MIN</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.000354</td>
<td>0.000002</td>
<td>0.000355</td>
<td>0.000213</td>
<td>0.000507</td>
<td>0.000021</td>
<td>0.000357</td>
<td>0.000398</td>
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<td>0.000373</td>
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<td>0.000376</td>
<td>0.000254</td>
<td>0.000640</td>
<td>0.000073</td>
<td>0.000383</td>
<td>0.000569</td>
</tr>
<tr>
<td>300</td>
<td>0.000391</td>
<td>0.000013</td>
<td>0.000395</td>
<td>0.000289</td>
<td>0.000745</td>
<td>0.000143</td>
<td>0.000412</td>
<td>0.000699</td>
</tr>
<tr>
<td>400</td>
<td>0.000407</td>
<td>0.000023</td>
<td>0.000414</td>
<td>0.000320</td>
<td>0.000829</td>
<td>0.000220</td>
<td>0.000444</td>
<td>0.000800</td>
</tr>
<tr>
<td>500</td>
<td>0.000422</td>
<td>0.000034</td>
<td>0.000431</td>
<td>0.000348</td>
<td>0.000898</td>
<td>0.000299</td>
<td>0.00048</td>
<td>0.000882</td>
</tr>
<tr>
<td>600</td>
<td>0.000436</td>
<td>0.000047</td>
<td>0.000448</td>
<td>0.000372</td>
<td>0.000955</td>
<td>0.000376</td>
<td>0.00052</td>
<td>0.000948</td>
</tr>
<tr>
<td>700</td>
<td>0.000448</td>
<td>0.000061</td>
<td>0.000464</td>
<td>0.000395</td>
<td>0.001003</td>
<td>0.000450</td>
<td>0.000565</td>
<td>0.001002</td>
</tr>
<tr>
<td>800</td>
<td>0.000460</td>
<td>0.000076</td>
<td>0.000480</td>
<td>0.000419</td>
<td>0.001044</td>
<td>0.000518</td>
<td>0.000613</td>
<td>0.001048</td>
</tr>
</tbody>
</table>

#### Table 3. Terminal failure rate as a function of time for β =2.

<table>
<thead>
<tr>
<th>Time (Hr)</th>
<th>SEN+</th>
<th>Pars network</th>
<th>Benes network</th>
<th>Two-layer MIN</th>
<th>SEN+</th>
<th>Pars network</th>
<th>Benes network</th>
<th>Two-layer MIN</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.000091</td>
<td>1.46E-11</td>
<td>0.000011</td>
<td>0.000006</td>
<td>0.000011</td>
<td>2.34E-10</td>
<td>0.000011</td>
<td>0.000006</td>
</tr>
<tr>
<td>200</td>
<td>0.000022</td>
<td>4.68E-10</td>
<td>0.000022</td>
<td>0.000011</td>
<td>0.000023</td>
<td>7.11E-10</td>
<td>0.000022</td>
<td>0.000012</td>
</tr>
<tr>
<td>300</td>
<td>0.000034</td>
<td>3.55E-9</td>
<td>0.000034</td>
<td>0.000017</td>
<td>0.000036</td>
<td>5.34E-8</td>
<td>0.000034</td>
<td>0.000021</td>
</tr>
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<td>0.000045</td>
<td>1.49E-8</td>
<td>0.000045</td>
<td>0.000024</td>
<td>0.000051</td>
<td>2.22E-7</td>
<td>0.000045</td>
<td>0.000031</td>
</tr>
<tr>
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<td>4.55E-8</td>
<td>0.000057</td>
<td>0.000031</td>
<td>0.000069</td>
<td>8.000001</td>
<td>0.000057</td>
<td>0.000045</td>
</tr>
<tr>
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<td>1.13E-7</td>
<td>0.000069</td>
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</tbody>
</table>

#### 3.2.2. Numerical results of the broadcast failure rate analysis

According to equations (25) through (28), the results of the broadcast failure rate analysis of 8×8 to 128×128 networks as a function of time for β =0.5, 1, and 2 are summarized in Tables 4 to 6, respectively. It should be mentioned that the network sizes in Table 4 have been chosen in such a way that no confusing result such as “undefined” is occurred. Undefined case illustrates a situation where reliability of the networks is almost equal to zero. Therefore, in these circumstances, the failure rate is undefined.
According to the results in Tables 4 to 6, it is evident that Pars network has the lowest broadcast failure rate compared with other networks for all scenarios for time, network size and the parameter \( \beta \). Therefore, these results reflect the fact that using several MINs in parallel has a higher potential than the other fault-tolerance methods of increasing the number of stages, concatenating a MIN with its reverse MIN, and using replicated MINs in terms of providing a lower broadcast failure rate.

### 3.2.3. Numerical results of the network failure rate analysis

According to equations (34) to (37), the results of the network failure rate analysis of \( 8 \times 8 \) to \( 128 \times 128 \) networks as a function of time for \( \beta = 0.5, 1 \), and 2 are summarized in Tables 7 to 9, respectively.

From Tables 7, 8, and 9, it is observed that the Pars network attains least network failure rate compared with the other networks for network size of 8. In addition, according to the results shown in these tables, for network size of 128, Pars has a better performance compared with SEN+ and two-layer MIN in terms of failure rate. However, as Tables 7 and 8 show, in network size of 128 for times 100 to 800 hours (Excluding time of 500 hours in Table 7), Benes network has a lower failure rate than the Pars and two other networks. Nevertheless, in Table 9, network size of 128 at times 100 to 500, Pars network outperforms Benes network in terms of failure rate. However, at 600 to 800 hours Benes network can again obtain better results in comparison with Pars.

In summary, based on Tables 7 through 9, in \( \beta = 0.5 \) and 1, Pars network achieves the best results for size of 8 and Benes network has the best results for size of 32 and 128. Also, in \( \beta = 2 \), it can be said that Pars achieves more acceptable performance compared with other networks for both sizes of 8 and 128.

In total, based on Tables 1 through 9, it can be concluded that the Pars network can be selected as a more appropriate network than the other networks in terms of failure rate. In other words, it can be concluded that the method of using several MINs in parallel has a good potential for designing networks with low failure rates.
failure rate of the networks will increase. Also, according to Tables 2, 4, and 7, it can be concluded that several MINs in parallel is suggested to yield lower failure rates compared to the other methods of increasing the number of stages, concatenating a MIN with its reverse MIN, and using replicated MINs.

4. Conclusion and Future Works

The methodology used in this analysis included three basic systematic phases: (1) Modeling the fault-tolerant MINs in complex series-parallel and non-series-parallel RBDs according to their topologies. (2) Deriving time-dependent reliability equations of the given RBDs. (3) Computing the failure rate (hazard function) of the networks according to their time-dependent reliability equations. Therefore, this methodology is rigorous and valuable for reliability analysis of other networks even for more complex structures. Numerical results obtained from this analysis confirmed that Pars network had an obvious advantage compared to other networks in terms of terminal, broadcast, and network failure rate for all operating times. Also, the supremacy of the Pars exists in all scenarios for the parameter \( \beta \) (0.5, 1, and 2) and for both network sizes 8 and 128. Thanks to the structure of Pars network, using several MINs in parallel is suggested to yield lower failure rates compared to the other methods of increasing the number of stages, concatenating a MIN with its reverse MIN, and using replicated MINs.

According to the numerical results, further useful analyses can be argued in terms of reliability engineering. The first issue here, is the effect of the parameter \( \beta \) (quality of switch failure rate) on the failure rate of the whole network. From the presented results in Tables 1, 4, and 7, it can be seen that value of \( \beta < 1 \) (\( \beta = 0.5 \)) can often reduce the failure rate of the networks over the time. Also, according to Tables 2, 4, and 7, it can be concluded that the value of \( \beta = 1 \) will result in increased or constant failure rate of the networks versus time. In addition, according to Tables 2, 4, and 7, it is clear that the rate of changes in the failure rate of the networks is slowly rises as a function of time. Finally, based on the Tables 3, 6, and 9, it is evident that, in case of \( \beta > 1 \) (\( \beta = 2 \)), as time increases, the failure rate of the networks will increase.

Regarding above discussion, it is obvious that the parameter \( \beta \) has a direct impact on the failure rate of networks. In other words, it is expected that the \( \beta < 1 \), \( \beta = 1 \) and \( \beta > 1 \) leads to reduced failure rates, almost constant

<table>
<thead>
<tr>
<th>Time (Hr)</th>
<th>SEN+ Pars network</th>
<th>Benes network</th>
<th>Two-layer MIN</th>
<th>SEN+ Pars network</th>
<th>Benes network</th>
<th>Two-layer MIN</th>
</tr>
</thead>
<tbody>
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</table>

<table>
<thead>
<tr>
<th>Time (Hr)</th>
<th>SEN+ Pars network</th>
<th>Benes network</th>
<th>Two-layer MIN</th>
<th>SEN+ Pars network</th>
<th>Benes network</th>
<th>Two-layer MIN</th>
</tr>
</thead>
<tbody>
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<td>0.001366</td>
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4. Conclusion and Future Works

The methodology used in this analysis included three basic systematic phases: (1) Modeling the fault-tolerant MINs in complex series-parallel and non-series-parallel RBDs according to their topologies. (2) Deriving time-dependent reliability equations of the given RBDs. (3) Computing the failure rate (hazard function) of the networks according to their time-dependent reliability equations. Therefore, this methodology is rigorous and valuable for reliability analysis of other networks even for more complex structures. Numerical results obtained from this analysis confirmed that Pars network had an obvious advantage compared to other networks in terms of terminal, broadcast, and network failure rate for all operating times. Also, the supremacy of the Pars exists in all scenarios for the parameter \( \beta \) (0.5, 1, and 2) and for both network sizes 8 and 128. Thanks to the structure of Pars network, using several MINs in parallel is suggested to yield lower failure rates compared to the other methods of increasing the number of stages, concatenating a MIN with its reverse MIN, and using replicated MINs.

According to the numerical results, further useful analyses can be argued in terms of reliability engineering. The first issue here, is the effect of the parameter \( \beta \) (quality of switch failure rate) on the failure rate of the whole network. From the presented results in Tables 1, 4, and 7, it can be seen that value of \( \beta < 1 \) (\( \beta = 0.5 \)) can often reduce the failure rate of the networks over the time. Also, according to Tables 2, 4, and 7, it can be concluded that the value of \( \beta = 1 \) will result in increased or constant failure rate of the networks versus time. In addition, according to Tables 2, 4, and 7, it is clear that the rate of changes in the failure rate of the networks is slowly rises as a function of time. Finally, based on the Tables 3, 6, and 9, it is evident that, in case of \( \beta > 1 \) (\( \beta = 2 \)), as time increases, the failure rate of the networks will increase.

Regarding above discussion, it is obvious that the parameter \( \beta \) has a direct impact on the failure rate of networks. In other words, it is expected that the \( \beta < 1 \), \( \beta = 1 \) and \( \beta > 1 \) leads to reduced failure rates, almost constant
failure rate, and increased failure rates over time, respectively. Although, these arguments are often true, but some situations might take place which affect the failure rate of a network. This is because the failure rate of a network is the result of dividing the pdf by the reliability function. Therefore, the failure rate is also dependent on the pdf and reliability changes over time. On the other hand, pdf and reliability are dependent on the structure of network, network size, and operating time. For instance, consider Pars network with network size = 8, β = 0.5, η = 6000, and γ = 0 in Table 1. For t = 100 hours, we have terminal failure rate = 0.000281 and for t = 200 hours, we have terminal failure rate = 0.000338. In this example, contrary to expectation, the failure rate at time 200 hours is not less than the failure rate at time 100 hours. In this case, it can be argued that the β < 1 can cause defective switching elements fail in the initial operating times of the networks. On the other hand, this initial time can be different for different networks. Therefore, before reaching this particular time, failure rate of the network can be increased. In Table 1, for network size of 8, as can be seen, failure rate of Pars network increases at times 100 to 700 hours. However, the failure rate of the network decreases when the time comes to 800 hours compared to the failure rate at the time of 700 hours. The results of Table 4 for 8×8 Pars network can be considered as another example of this case.

The next issue is about the impact of network size on the failure rate. In the most cases, increasing the network size can lead to increased failure rate. The reason is that increasing the network size causes the network complexity increases, which in turn leads to reduction of reliability and increase of pdf. However, in order to increase the failure rate, the decrease in reliability must be greater than the rise in pdf or vice versa. Otherwise, it is possible that the failure rate remains constant or even decreases versus network size. This is because the failure rate of a network is the result of dividing the pdf by the reliability function.

Future works can be divided into two bunches: (1) Designing high-performance networks; conducted analyzes in this paper can be a good guide in the design of high-performance networks. Moreover, regarding this study that indicated using several MINs in parallel can significantly improve fault-tolerance capability, another innovative approach in the future could be a combination of the methods. (2) Reliability engineering; the reliability equations calculated in this paper was used to obtain the failure rate of the networks. However, some of the other reliability parameters such as mean time to failure (MTTF), conditional reliability, and warranty period and BX life can be calculated to study other aspects of the behavior of the networks.

Appendix

In equation (17), each of the equations E1 to E10 are calculated as follows:

\[ E1 = \frac{\beta \cdot (\log_2(N))^\beta \cdot \left(\frac{1}{\eta} \cdot \frac{1}{\log_2(N)}\right)^{\beta+1} \cdot \left(\frac{1}{\eta} \cdot \frac{1}{\log_2(N)}\right)^{\gamma+1}}{\eta \cdot (\log_2(N))^\beta \cdot \eta^\gamma} \]  

\[ E2 = \frac{2\beta \cdot (\log_2(N))^\beta \cdot \left(\frac{1}{\eta} \cdot \frac{1}{\log_2(N)}\right)^{\beta+1} \cdot \left(\frac{1}{\eta} \cdot \frac{1}{\log_2(N)}\right)^{\gamma+1} \cdot \left(\frac{1}{\eta} \cdot \frac{1}{\log_2(N)}\right)^{\gamma+1}}{\eta \cdot (\log_2(N))^\beta \cdot \eta^\gamma} \]  

\[ E3 = \frac{2\beta \cdot (\log_2(N))^\beta \cdot \left(\frac{1}{\eta} \cdot \frac{1}{\log_2(N)}\right)^{\beta+1} \cdot \left(\frac{1}{\eta} \cdot \frac{1}{\log_2(N)}\right)^{\gamma+1} \cdot \left(\frac{1}{\eta} \cdot \frac{1}{\log_2(N)}\right)^{\gamma+1}}{\eta \cdot (\log_2(N))^\beta \cdot \eta^\gamma} \]  

\[ E4 = \frac{2\beta \cdot (\log_2(N))^\beta \cdot \left(\frac{1}{\eta} \cdot \frac{1}{\log_2(N)}\right)^{\beta+1} \cdot \left(\frac{1}{\eta} \cdot \frac{1}{\log_2(N)}\right)^{\gamma+1} \cdot \left(\frac{1}{\eta} \cdot \frac{1}{\log_2(N)}\right)^{\gamma+1}}{\eta \cdot (\log_2(N))^\beta \cdot \eta^\gamma} \]  

\[ E5 = \frac{\beta \cdot (\log_2(N))^\beta \cdot \left(\frac{1}{\eta} \cdot \frac{1}{\log_2(N)}\right)^{\beta+1} \cdot \left(\frac{1}{\eta} \cdot \frac{1}{\log_2(N)}\right)^{\gamma+1} \cdot \left(\frac{1}{\eta} \cdot \frac{1}{\log_2(N)}\right)^{\gamma+1}}{\eta \cdot (\log_2(N))^\beta \cdot \eta^\gamma} \]
The equations E6 to E10 are as follows:

\[ E6 = \frac{2\beta(\frac{\gamma}{\gamma-1})^{-\beta-1}}{\eta e^{(\frac{\gamma}{\gamma-1})^\beta}} \left( \frac{1}{\gamma-1} \right)^{\frac{\gamma}{\gamma-1}} \left( \frac{1}{\gamma-1} \right)^{\frac{\gamma}{\gamma-1}} e^{(\frac{\gamma}{\gamma-1})^\beta} \]  

\[ E7 = \frac{2\beta(\log_\gamma(N)-2)^{-\beta-1}}{\eta e^{(\frac{\gamma}{\gamma-1})^\beta}} \left( \frac{1}{\gamma-1} \right)^{\frac{\gamma}{\gamma-1}} \left( \frac{1}{\gamma-1} \right)^{\frac{\gamma}{\gamma-1}} e^{(\frac{\gamma}{\gamma-1})^\beta} \]  

\[ E8 = \frac{2\beta(\frac{\gamma}{\gamma-1})^{-\beta-1}}{\eta e^{(\frac{\gamma}{\gamma-1})^\beta}} \left( \frac{1}{\gamma-1} \right)^{\frac{\gamma}{\gamma-1}} \left( \frac{1}{\gamma-1} \right)^{\frac{\gamma}{\gamma-1}} e^{(\frac{\gamma}{\gamma-1})^\beta} \]  

\[ E9 = \frac{2\beta(\frac{\gamma}{\gamma-1})^{-\beta-1}}{\eta e^{(\frac{\gamma}{\gamma-1})^\beta}} \left( \frac{1}{\gamma-1} \right)^{\frac{\gamma}{\gamma-1}} \left( \frac{1}{\gamma-1} \right)^{\frac{\gamma}{\gamma-1}} e^{(\frac{\gamma}{\gamma-1})^\beta} \]  

\[ E10 = \frac{2\beta(\frac{\gamma}{\gamma-1})^{-\beta-1}}{\eta e^{(\frac{\gamma}{\gamma-1})^\beta}} \left( \frac{1}{\gamma-1} \right)^{\frac{\gamma}{\gamma-1}} \left( \frac{1}{\gamma-1} \right)^{\frac{\gamma}{\gamma-1}} e^{(\frac{\gamma}{\gamma-1})^\beta} \]  

In the equation (26), each of the equations E1 to E8 are calculated as follows:

\[ E1 = \frac{2\beta(\frac{\gamma}{\gamma-1})^{-\beta-1}}{\eta e^{(\frac{\gamma}{\gamma-1})^\beta}} \left( \frac{1}{\gamma-1} \right)^{\frac{\gamma}{\gamma-1}} \left( \frac{1}{\gamma-1} \right)^{\frac{\gamma}{\gamma-1}} e^{(\frac{\gamma}{\gamma-1})^\beta} \]  

\[ E2 = \left(1-e^{-\frac{3\gamma}{\gamma-1}}(\frac{\gamma}{\gamma-1})^\beta\right)^2 \left(1-e^{-\frac{3\gamma}{\gamma-1}}(\frac{\gamma}{\gamma-1})^\beta\right) \left(1-e^{-\frac{3\gamma}{\gamma-1}}(\frac{\gamma}{\gamma-1})^\beta\right) \left(1-e^{-\frac{3\gamma}{\gamma-1}}(\frac{\gamma}{\gamma-1})^\beta\right) \]  

\[ E3 = \frac{2\beta(\frac{\gamma}{\gamma-1})^{-\beta-1}}{\eta e^{(\frac{\gamma}{\gamma-1})^\beta}} \left( \frac{1}{\gamma-1} \right)^{\frac{\gamma}{\gamma-1}} \left( \frac{1}{\gamma-1} \right)^{\frac{\gamma}{\gamma-1}} e^{(\frac{\gamma}{\gamma-1})^\beta} \]  

\[ E4 = \frac{2\beta(\frac{\gamma}{\gamma-1})^{-\beta-1}}{\eta e^{(\frac{\gamma}{\gamma-1})^\beta}} \left( \frac{1}{\gamma-1} \right)^{\frac{\gamma}{\gamma-1}} \left( \frac{1}{\gamma-1} \right)^{\frac{\gamma}{\gamma-1}} e^{(\frac{\gamma}{\gamma-1})^\beta} \]  

\[ E5 = \frac{2\beta(\frac{\gamma}{\gamma-1})^{-\beta-1}}{\eta e^{(\frac{\gamma}{\gamma-1})^\beta}} \left( \frac{1}{\gamma-1} \right)^{\frac{\gamma}{\gamma-1}} \left( \frac{1}{\gamma-1} \right)^{\frac{\gamma}{\gamma-1}} e^{(\frac{\gamma}{\gamma-1})^\beta} \]  

\[ E6 = \frac{2\beta(\frac{\gamma}{\gamma-1})^{-\beta-1}}{\eta e^{(\frac{\gamma}{\gamma-1})^\beta}} \left( \frac{1}{\gamma-1} \right)^{\frac{\gamma}{\gamma-1}} \left( \frac{1}{\gamma-1} \right)^{\frac{\gamma}{\gamma-1}} e^{(\frac{\gamma}{\gamma-1})^\beta} \]  

\[ E7 = \frac{2\beta(\frac{\gamma}{\gamma-1})^{-\beta-1}}{\eta e^{(\frac{\gamma}{\gamma-1})^\beta}} \left( \frac{1}{\gamma-1} \right)^{\frac{\gamma}{\gamma-1}} \left( \frac{1}{\gamma-1} \right)^{\frac{\gamma}{\gamma-1}} e^{(\frac{\gamma}{\gamma-1})^\beta} \]
In equation (34), equations E1 to E3 are computed as follows:

\[ E_1 = \frac{\frac{1}{\sqrt{4\pi \eta \left[ \log_2 (N) \right]}}}{} \]

\[ E_2 = \frac{\frac{1}{\sqrt{4\pi \eta \left[ \log_2 (N) \right] \left[ \log_2 (N) \right]}}}{} \]

\[ E_3 = \frac{\frac{1}{\sqrt{4\pi \eta \left[ \log_2 (N-1) \right] \left[ \log_2 (N-1) \right]}}}{} \]

In equation (35), equations E1 to E12 are computed as follows:

\[ E_1 = \frac{\frac{1}{\sqrt{4\pi \eta \left[ \log_2 (N-1) \right] \left[ \log_2 (N) \right] \left[ \log_2 (N) \right]}}}{} \]

\[ E_2 = \frac{\frac{1}{\sqrt{4\pi \eta \left[ \log_2 (N-1) \right] \left[ \log_2 (N) \right] \left[ \log_2 (N) \right]}}}{} \]

\[ E_3 = \frac{\frac{1}{\sqrt{4\pi \eta \left[ \log_2 (N-1) \right] \left[ \log_2 (N) \right] \left[ \log_2 (N) \right]}}}{} \]

\[ E_4 = \frac{\frac{1}{\sqrt{4\pi \eta \left[ \log_2 (N-1) \right] \left[ \log_2 (N) \right] \left[ \log_2 (N) \right]}}}{} \]

\[ E_5 = \frac{\frac{1}{\sqrt{4\pi \eta \left[ \log_2 (N-1) \right] \left[ \log_2 (N) \right] \left[ \log_2 (N) \right]}}}{} \]

\[ E_6 = \frac{\frac{1}{\sqrt{4\pi \eta \left[ \log_2 (N-1) \right] \left[ \log_2 (N) \right] \left[ \log_2 (N) \right]}}}{} \]

\[ E_7 = \frac{\frac{1}{\sqrt{4\pi \eta \left[ \log_2 (N-1) \right] \left[ \log_2 (N) \right] \left[ \log_2 (N) \right]}}}{} \]

\[ E_8 = \frac{\frac{1}{\sqrt{4\pi \eta \left[ \log_2 (N-1) \right] \left[ \log_2 (N) \right] \left[ \log_2 (N) \right]}}}{} \]
\[ E_9 = \left( 1 - e^{-\frac{3N(t-\gamma)}{n}} \right) \left( 1 - e^{-\frac{\beta}{n}} \right) \left( 1 - e^{-\frac{\beta}{n}} \right) + 1 \]  

\[ E_{10} = \left( 1 - e^{-\frac{3N(t-\gamma)}{n}} \right) \left( 1 - e^{-\frac{\beta}{n}} \right) \left( 1 - e^{-\frac{\beta}{n}} \right) + 1 \]  

\[ E_{11} = e^{\frac{3N(t-\gamma)^2}{n}} \left( 1 - e^{\frac{\beta}{n}} \right) \left( 1 - e^{\frac{\beta}{n}} \right) + 1 \]  

\[ E_{12} = \left( 1 - e^{-\frac{3N(t-\gamma)^2}{n}} \right) \left( 1 - e^{\frac{\beta}{n}} \right) \left( 1 - e^{\frac{\beta}{n}} \right) + 1 \]  

In equation (37), equations E1 to E3 are computed as follows:

\[ E_1 = \frac{\beta^P}{\eta^p} \left( \frac{1}{(\frac{1}{\gamma} + \frac{1}{t-\gamma})} + 1 \right) \left( \frac{1}{(\frac{1}{\gamma} + \frac{1}{t-\gamma})} + 1 \right) \left( \frac{1}{(\frac{1}{\gamma} + \frac{1}{t-\gamma})} + 1 \right) \]  

\[ E_2 = \frac{\beta^P}{\eta^p} \left( \frac{1}{(\frac{1}{\gamma} + \frac{1}{t-\gamma})} + 1 \right) \left( \frac{1}{(\frac{1}{\gamma} + \frac{1}{t-\gamma})} + 1 \right) \left( \frac{1}{(\frac{1}{\gamma} + \frac{1}{t-\gamma})} + 1 \right) \]  

\[ E_3 = \frac{\beta^P}{\eta^p} \left( \frac{1}{(\frac{1}{\gamma} + \frac{1}{t-\gamma})} + 1 \right) \left( \frac{1}{(\frac{1}{\gamma} + \frac{1}{t-\gamma})} + 1 \right) \left( \frac{1}{(\frac{1}{\gamma} + \frac{1}{t-\gamma})} + 1 \right) \]  

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**References**


Figures

(a) 8×8 SEN+, (b) 8×8 Pars network, (c) 8×8 Benes network, and (d) 8×8 two-layer replicated MIN.

Fig. 1. (a) 8×8 SEN+, (b) 8×8 Pars network, (c) 8×8 Benes network, and (d) 8×8 two-layer replicated MIN.

Fig. 2. Terminal reliability RBD for N×N SEN+ [19].

Fig. 3. Terminal reliability RBD for N×N Pars network.

Fig. 4. Simplified terminal reliability RBD for N×N Pars network.
Fig. 5. Terminal reliability RBD of $N \times N$ Pars network for equation $R_{key} = R_{Pars} \cdot R_{key}$. 

Fig. 6. Terminal reliability RBD for (a) $8 \times 8$ Benes network and (b) $16 \times 16$ Benes network. 

Fig. 7. Terminal reliability RBD for $N \times N$ two-layer replicated MIN. 

Fig. 8. Broadcast reliability RBD for $N \times N$ SEN+ [19]. 

Fig. 9. Broadcast reliability RBD for $N \times N$ Pars network. 

Fig. 10. Broadcast reliability RBD for (a) $8 \times 8$ Benes network and (b) $16 \times 16$ Benes network.
Fig. 11. Broadcast reliability RBD for $N \times N$ two-layer replicated MIN.

Fig. 12. Network reliability RBD for $N \times N$ SEN+ [19].

Fig. 13. Network reliability RBD for $N \times N$ Pars network.

Fig. 14. Network reliability RBD for $N \times N$ two-layer MIN.

Fig. 15. Network reliability RBD for (a) $8 \times 8$ Benes network and (b) $16 \times 16$ Benes network.

Fig. 16. The Weibull pdf for different values of $\eta$ [52].